

A NEW RENORMALIZATION GROUP FOR HAMILTONIAN FIELD THEORY

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The Schrödinger equation with a two-dimensional δ -function potential is a simple example of an asymptotically free theory that undergoes dimensional transmutation. Renormalization requires the introduction of a mass scale, which can be lowered perturbatively until an infrared cutoff produced by non-perturbative effects such as bound state formation is encountered. We outline the effective field theory and similarity renormalization group techniques for producing renormalized cutoff hamiltonians, and illustrate the control of logarithmic and inverse-power-law errors both techniques provide.

1 Motivation and Outline

This article follows a more complete discussion in a longer article being prepared for publication.¹ Elsewhere in these Proceedings the possibility of deriving a valid constituent approximation in light-front QCD is discussed.² A simple procedure that produces a renormalized light-front QCD hamiltonian starts with the computation of a cutoff hamiltonian using a similarity renormalization group (SRG)^{3,4,5,6} and coupling coherence.^{7,8,9} In this article we illustrate this step using the two-dimensional δ -function in a one-body Schrödinger equation.¹⁰ We also discuss a related approach, effective field theory (EFT).^{11,12}

Locality is the source of perturbative divergences in field theory, and it produces similar effects in simple one-body quantum mechanics problems. Locality leads us to consider potentials consisting of δ -functions and their derivatives. These potentials produce divergences that require the introduction of a mass scale. Changing this mass scale should be equivalent to changing a dimensionless coupling constant (*i.e.*, dimensional transmutation) for a scale-invariant local hamiltonian. The two-dimensional δ -function problem is solved in the second section, providing the *data* for our error analysis of EFT and SRG calculations.

Problems requiring renormalization result from an arbitrarily large number of degrees of freedom being coupled. This is clear in momentum space, where the off-diagonal matrix elements of a δ -function are constants. Divergences result from an infinite number of scales being directly coupled, and renormalization requires a cutoff λ to regulate the divergences. Physical re-

sults such as binding energies should be independent of λ , so there should exist a line of cutoff hamiltonians H_λ that all produce the same physical results. A renormalization group (RG) is built from transformations, $\mathcal{T}(\lambda, \lambda')$, that change the cutoff. \mathcal{T} connects hamiltonians at different scales and in the SRG \mathcal{T} is typically a unitary transformation.

In EFT \mathcal{T} exists in principle, but H_λ is determined at each scale by constraining the couplings in front of a finite number of local operators to fit data. A simple scale transformation leads to the classification of these operators as relevant (powers of the cutoff appear), marginal (the cutoff appears only in logarithms that are absorbed in the running marginal coupling), and irrelevant (inverse powers of the cutoff appear). The same operators arise in the SRG, where their strengths are approximated by expansions in powers of a running marginal coupling. For the simple two-dimensional δ -function there are no relevant operators and only one marginal operator. There are always an infinite number of irrelevant operators, and these become increasingly important as the cutoff approaches a non-perturbative energy scale in the problem.

The easiest way to clarify these points is by example. We show a complete solution of the two-dimensional δ -function problem, introducing the K-matrix to compute scattering phase shifts. There is one bound state, and its binding energy sets the scale for non-perturbative physics. The EFT hamiltonian is produced by truncating the series of irrelevant operators, and then fitting a finite number of remaining couplings to an equal number of low-energy scattering constraints. This leads to power-law suppression of errors in the predicted binding energy, with the expansion clearly breaking down as the cutoff approaches the binding energy. The SRG hamiltonian is produced by solving a differential equation for H_λ subject to the constraint that all irrelevant couplings are analytic functions of the single marginal coupling.

2 The two-dimensional δ -function potential

Consider the Schrödinger equation in two dimensions with an attractive Dirac δ -function potential:¹⁰

$$-\nabla_{\mathbf{r}}^2 \Psi(\mathbf{r}) - \alpha_0 \delta^{(2)}(\mathbf{r}) \Psi(\mathbf{r}) = E \Psi(\mathbf{r}) . \quad (1)$$

The coupling α_0 is dimensionless, so the hamiltonian is scale invariant (i.e., there is no intrinsic length or energy scale). It is relatively straightforward to solve this equation in position space by introducing a convenient distribution

function to regulate the δ -function, but we will work in momentum space to stay close to the EFT and SRG calculations.

The Schrödinger equation in momentum space is,

$$p^2 \Phi(\mathbf{p}) - \frac{\alpha_0}{(2\pi)^2} \int d^2 q \Phi(\mathbf{q}) = E \Phi(\mathbf{p}) . \quad (2)$$

As a consequence of scale invariance, if there is a bound state solution to Eq.(2) then it will admit solutions for any $E < 0$. This corresponds to a continuum of bound states with energies extending down to $-\infty$, so the system is not bounded from below. By rearranging the terms in the Schrödinger equation we obtain

$$\Phi(\mathbf{p}) = \frac{\alpha_0}{2\pi} \frac{\Psi(0)}{(p^2 + E_0)} , \quad (3)$$

where $\Psi(0)$ is the position space wave-function at the origin and $E_0 = -E > 0$ is the binding energy.

To obtain the eigenvalue condition for the binding energy, we can integrate both sides of Eq. (3):

$$1 = \frac{\alpha_0}{2\pi} \int_0^\infty dp p \frac{1}{(p^2 + E_0)} . \quad (4)$$

The integral on the r.h.s. diverges logarithmically, so the problem is ill-defined.

Renormalization requires that we regulate the divergence and absorb cut-off dependence in the coupling. First, we regulate the integral with a momentum cutoff, obtaining

$$1 = \frac{\alpha_0}{2\pi} \int_0^\Lambda dp p \frac{1}{(p^2 + E_0)} = \frac{\alpha_0}{4\pi} \ln \left(1 + \frac{\Lambda^2}{E_0} \right) , \quad (5)$$

so that

$$E_0 = \frac{\Lambda^2}{e^{-\frac{4\pi}{\alpha_0}} - 1} . \quad (6)$$

Clearly, if the coupling α_0 is fixed then $E_0 \rightarrow \infty$ as $\Lambda \rightarrow \infty$. In order to eliminate the divergence and produce a finite, well-defined bound state we can renormalize the theory by demanding that the coupling runs with the cutoff Λ in such a way that the binding energy remains fixed as the cutoff is removed:

$$\alpha_0 \rightarrow \alpha_\Lambda = \frac{4\pi}{\ln \left(1 + \frac{\Lambda^2}{E_0} \right)} . \quad (7)$$

The dimensionless renormalized running coupling α_Λ that characterizes the strength of the interaction is therefore replaced by a new (dimensionful) parameter E_0 , the binding energy of the system. This is a simple example of dimensional transmutation: even though the original “bare” hamiltonian is scale invariant, renormalization leads to a scale that characterizes the physical observables. Note that E_0 can be chosen arbitrarily, fixing the energy scale of the underlying (renormalized) theory. It is also interesting to note that the renormalized running coupling α_Λ vanishes as $\Lambda \rightarrow \infty$ and so the theory is asymptotically free.

This renormalized hamiltonian can be used to compute other observables. The usual prescription for the calculations is to obtain the solutions with the cutoff in place and then let it approach ∞ , using the running coupling in Eq. 7. If an exact calculation can be implemented, the final results should be independent of the regularization and renormalization schemes. As an example, we calculate the scattering wave function,

$$\Phi_k(\mathbf{p}) = \delta^{(2)}(\mathbf{p} - \mathbf{k}) + \frac{\alpha_\Lambda}{2\pi} \frac{\Psi(0)}{(p^2 - k^2 + i\epsilon)}, \quad (8)$$

where $k = \sqrt{E}$. Integrating both sides over \mathbf{p} with a cutoff Λ in place, we obtain

$$\Psi(0) = \frac{1}{2\pi} \left[1 - \frac{\alpha_\Lambda}{4\pi} \ln \left(1 + \frac{\Lambda^2}{-k^2 + i\epsilon} \right) \right]^{-1}; \quad (9)$$

thus,

$$\alpha_\Lambda \Psi(0) = \frac{1}{2\pi} \left[\frac{1}{4\pi} \ln \left(1 + \frac{\Lambda^2}{E_0} \right) - \frac{1}{4\pi} \ln \left(1 + \frac{\Lambda^2}{-k^2 + i\epsilon} \right) \right]^{-1}. \quad (10)$$

In the limit $\Lambda \rightarrow \infty$ we obtain:

$$\alpha_\Lambda \Psi(0) = \frac{2}{\ln \left(\frac{k^2}{E_0} \right) - i\pi}. \quad (11)$$

The resulting scattering wavefunction is then given by

$$\Phi_k(\mathbf{p}) = \delta^{(2)}(\mathbf{p} - \mathbf{k}) + \frac{1}{2\pi} \frac{2}{(p^2 - k^2)} \left[\ln \left(\frac{k^2}{E_0} \right) - i\pi \right]^{-1}. \quad (12)$$

Only S-wave scattering occurs, because the centrifugal barrier completely screens the δ -function potential for higher angular momentum.

For our numerical calculations we used the K-matrix, which has mixed boundary conditions that lead to a principal value. The K-matrix Lippmann-Schwinger equation with the renormalized potential is given by:

$$K(\mathbf{p}, \mathbf{p}'; k) = V(\mathbf{p}, \mathbf{p}') + \mathcal{P} \int d^2 q \frac{V(\mathbf{p}, \mathbf{q})}{k^2 - q^2} K(\mathbf{q}, \mathbf{p}'; k). \quad (13)$$

Since only S-wave scattering takes place we can integrate over the angular variable, obtaining

$$K^{(l=0)}(p, p'; k) = V^{(l=0)}(p, p') + \mathcal{P} \int_0^\Lambda dq q \frac{V^{(l=0)}(p, q)}{k^2 - q^2} K^{(l=0)}(q, p'; k), \quad (14)$$

where

$$V^{(l=0)}(p, p') = -\frac{\alpha_\Lambda}{2\pi}. \quad (15)$$

The Lippmann-Schwinger equation for the “on-shell” K-matrix is given by:

$$K^{(l=0)}(k) = -\frac{\alpha_\Lambda}{2\pi} - \frac{\alpha_\Lambda}{2\pi} K^{(l=0)}(k) \mathcal{P} \int_0^\Lambda dq q \frac{1}{k^2 - q^2}. \quad (16)$$

Solving this equation and taking the limit $\Lambda \rightarrow \infty$, we obtain the exact on-shell K-matrix:

$$K_0(k) = -\frac{2}{\ln\left(\frac{k^2}{E_0}\right)}. \quad (17)$$

Using

$$k \cot \delta_0(k) = -\frac{2k}{\pi} \frac{1}{K_0(k)}, \quad (18)$$

we can obtain the exact phase-shifts:

$$\cot \delta_0 = \frac{1}{\pi} \ln\left(\frac{k^2}{E_0}\right). \quad (19)$$

3 Effective field theory

3.1 EFT formalism

EFT provides a systematic procedure for improving the approximate description of composite effective degrees of freedom and their dynamics by increasing the number of local effective interactions. The effects of high-energy degrees of

freedom are incorporated in effective interactions whose strength is adjusted to fit appropriate low-energy results. The keys to maintaining predictive power in EFT are approximate locality and an expansion in powers of a small ratio of low-to-high energy scales or equivalently, short-to-long distance scales.

We follow Lepage's treatment of EFT.¹¹ For simplicity we use a separable hamiltonian with a cutoff λ that smoothly regulates interactions involving high energy states:

$$V_\lambda(p, p') = \left[C_0(\lambda) + C_2(\lambda) \frac{(p^2 + p'^2)}{2\lambda^2} + C_4(\lambda) \frac{(p^4 + p'^4)}{4\lambda^4} + C_4'(\lambda) \frac{p^2 p'^2}{2\lambda^4} + \dots \right] \times e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}}. \quad (20)$$

The parameters C_i in the expansion can be determined by fitting data for low-energy processes. Here we use as *data* the values for the inverse on-shell K-matrix, calculated with the exact theory. We follow the method described by Steele and Furnstahl¹³ with the parameters fixed so that EFT gives the same momentum expansion for the inverse on-shell K-matrix as the exact theory to a given order.

The Lippmann-Schwinger equation for the K-matrix with is

$$K_\lambda(p, p'; k) = V_\lambda(p, p') + \mathcal{P} \int d^2q \frac{V_\lambda(p, q)}{k^2 - q^2} K_\lambda(q, p'; k), \quad (21)$$

where \mathcal{P} denotes the principal value. The equation refers to the S-wave potential (angular variable integrated out) and $d^2q = q dq$. To obtain the K-matrix we solve the Lippmann-Schwinger equation nonperturbatively.^{14,15,16} The effective potential given in Eq.(20) (truncated at order p^4) can be written in the form

$$V_\lambda(p, p') = e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}} \sum_{i,j=0}^2 p^{2i} \Lambda_{ij} p'^{2j}, \quad (22)$$

where Λ_{ij} are the matrix elements of

$$\mathbf{\Lambda} = \begin{pmatrix} C_0(\lambda) & C_2(\lambda)/2\lambda^2 & C_4(\lambda)/4\lambda^4 \\ C_2(\lambda)/2\lambda^2 & C_4'(\lambda)/2\lambda^4 & 0 \\ C_4(\lambda)/4\lambda^4 & 0 & 0 \end{pmatrix}. \quad (23)$$

The solution of the Lippmann-Schwinger equation can then be written in the form

$$K_\lambda(p', p; k) = e^{-\frac{p^2}{2\lambda^2}} e^{-\frac{p'^2}{2\lambda^2}} \sum_{i,j=0}^1 p^{2i} \tau_{ij}(k) p'^{2j} . \quad (24)$$

The unknown matrix $\tau(k)$ satisfies the equation

$$\tau(\mathbf{k}) = \mathbf{\Lambda} + \mathbf{\Lambda} \mathcal{I}(k) \tau(\mathbf{k}) , \quad (25)$$

where

$$\mathcal{I} = \begin{pmatrix} I_0(k) & I_1(k) & I_2(k) \\ I_1(k) & I_2(k) & I_3(k) \\ I_2(k) & I_3(k) & I_4(k) \end{pmatrix} , \quad (26)$$

with

$$I_n(k) = \mathcal{P} \int dq q^{2n+D-1} \frac{1}{k^2 - q^2} e^{-\frac{q^2}{\lambda^2}} . \quad (27)$$

Solving Eq. (25) for τ analytically and substituting the solution in Eq. (24) we obtain the effective K-matrix.

It is straightforward to extend the calculation to higher orders. We fit the difference between the effective and exact inverse on-shell K-matrix ($p^2 = p'^2 = k^2$) to an interpolating polynomial in k^2/λ^2 to the highest possible order,

$$\Delta \left[\frac{1}{K} \right] = A_0 + A_2 \frac{k^2}{\lambda^2} + A_4 \frac{k^4}{\lambda^4} + \dots . \quad (28)$$

The coefficients A_i are minimized with respect to the variation in the parameters C_i of the effective potential. The number of coefficients that can be minimized is given by the number of parameters appearing in the effective potential.

By adjusting only the parameter $C_0(\lambda)$ we should eliminate the leading error. As each term is added to the effective potential, followed by the adjustment of the respective parameter, we expect the errors in the on-shell K-matrix to be systematically reduced by *powers of k^2/λ^2* . Thus, in a log-log plot for $\Delta[1/K]$ we expect to obtain straight lines with slope given by the dominant power of k/λ in the error.

The expansion given by Eq.(28) becomes invalid at best when the momenta involved are of the same order as the cutoff, where one expects the short distance effects to be directly resolved. This point corresponds to the radius

of convergence of the effective theory. Using the effective potential (truncated at a given order) with the parameters fixed by fitting the inverse on-shell K-matrix we diagonalize the effective hamiltonian numerically, obtaining a bound-state energy that can be compared to the exact result.

3.2 EFT results for the two-dimension δ -function

The EFT power counting scheme is simple. For the leading-order prediction, we use the effective potential with one parameter. Evaluating the difference $\Delta [1/K^{(0)}] = 1/K_\lambda^{(0)} - 1/K_0$ and expanding it in powers of k/λ we obtain:

$$\Delta \left[\frac{1}{K^{(0)}} \right] = \left[\frac{1}{C_0} - \frac{\gamma}{2} + \frac{1}{2} \ln \left(\frac{\lambda^2}{E_0} \right) \right] + \left[\frac{1}{C_0} - \frac{1}{2} \right] \frac{k^2}{\lambda^2} + \mathcal{O} \left(\frac{k^4}{\lambda^4} \right). \quad (29)$$

By choosing

$$C_0(\lambda) = \frac{-2}{\ln \left(\frac{\lambda^2}{E_0} \right) - \gamma}, \quad (30)$$

the leading logarithmic error is eliminated and the remaining errors are dominated by the term

$$\mathcal{O} \left(\frac{k^2}{\lambda^2} \right) = \frac{1}{2} \left[\gamma - \ln \left(\frac{\lambda^2}{E_0} \right) - 1 \right] \frac{k^2}{\lambda^2}. \quad (31)$$

To evaluate the errors in the binding energy we use the effective potential with one, two and four parameters adjusted to fit the inverse K-matrix. We solve the Schrödinger equation numerically for different values of the cutoff λ . In Fig. 1 we show in a log-log plot the absolute values for the relative errors in the bound-state energy as a function of E_0/λ^2 . As expected, we see straight lines with slope given by the dominant power of E_0/λ^2 in the error. Note that the dominant logarithmic errors are completely absorbed by adjusting the parameter C_0 and when more parameters are adjusted we obtain power-law improvement.

4 The similarity renormalization group

4.1 SRG formalism

The similarity renormalization group^{3,4} is made of similarity transformations that run a smooth cutoff on the magnitude of off-diagonal matrix elements. As the cutoff is lowered, the hamiltonian is forced toward the diagonal. It is straightforward to show that an infinitesimal transformation produces a

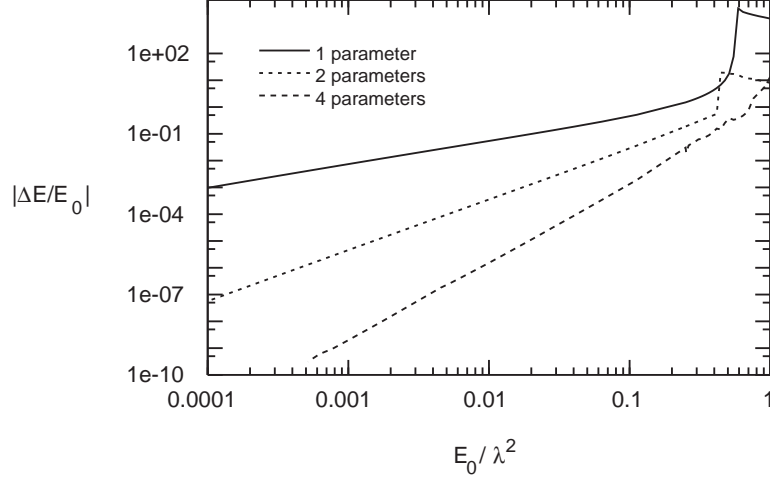


Figure 1. The EFT error in the binding energy for the two-dimensional delta-function using one, two and four parameters. The exact theory is fixed by choosing $E_0 = 1$.

differential equation governing the evolution of the hamiltonian. Głazek and Wilson considered very general transformations in their original work,³ but we use the simple transformation developed by Wegner.⁴ The infinitesimal transformation is

$$\frac{dH_s}{ds} = [H_s, [H_s, h]] . \quad (32)$$

Here $H_s = h + V_s$, where h is the free hamiltonian, evolves with a flow parameter s that ranges from 0 to ∞ . The flow-parameter has dimensions $1/(\text{energy})^2$ and can be expressed in terms of the similarity cutoff λ by the relation $s = 1/\lambda^2$. Wegner advocates using the full diagonal part of the hamiltonian instead of using h in the transformation.

The reduced interaction, \overline{V}_{sij} (the interaction with a gaussian similarity cutoff factored out) is defined by

$$V_{sij} = e^{-s\Delta_{ij}^2} \overline{V}_{sij} , \quad (33)$$

where $\Delta_{ij} = p_i^2 - p_j^2$ for our simple transformation. The gaussian factor forces the interaction toward the diagonal to leading order, and the equation for the evolution of \overline{V} starts at second order.

The flow equation for the reduced interaction is

$$\frac{d\bar{V}_{sij}}{ds} = \sum_k (\Delta_{ik} + \Delta_{jk}) \bar{V}_{sik} \bar{V}_{skj} e^{-2s\Delta_{ik}\Delta_{jk}} , \quad (34)$$

where we use $\Delta_{ij}^2 - \Delta_{ik}^2 - \Delta_{jk}^2 = -2\Delta_{ik}\Delta_{jk}$.

To solve this equation we impose a boundary condition, $H_s|_{s \rightarrow s_0} \equiv H_{s_0}$, where s_0 can be expressed in terms of the large similarity cutoff as $s_0 = 1/\Lambda^2$. Next we make a perturbative expansion,

$$\bar{V}_s = \bar{V}_s^{(1)} + \bar{V}_s^{(2)} + \dots , \quad (35)$$

where the superscript implies the order in the running canonical coupling. All counterterms are determined using coupling coherence, which assumes \bar{V} is a perturbative function of α .

At first order we have

$$\frac{d\bar{V}_{sij}^{(1)}}{ds} = 0 , \quad (36)$$

which implies

$$\bar{V}_{sij}^{(1)} = \bar{V}_{s_0 ij} , \quad (37)$$

where s is the final scale. Because of the dimensions of the flow parameter we have $s > s_0$, corresponding to a smaller cutoff. The “no cutoff limit” corresponds to $s_0 \rightarrow 0$.

At second order we have

$$\frac{d\bar{V}_{sij}^{(2)}}{ds} = \sum_k (\Delta_{ik} + \Delta_{jk}) \bar{V}_{s_0 ik} \bar{V}_{s_0 kj} e^{-2s\Delta_{ik}\Delta_{jk}} . \quad (38)$$

Integrating, we obtain

$$\begin{aligned} \bar{V}_{sij}^{(2)} = \frac{1}{2} \sum_k \bar{V}_{s_0 ik} \bar{V}_{s_0 kj} \left(\frac{1}{\Delta_{ik}} + \frac{1}{\Delta_{jk}} \right) \times \\ \times \left[e^{-2s\Delta_{ik}\Delta_{jk}} - e^{-2s_0\Delta_{ik}\Delta_{jk}} \right] . \end{aligned} \quad (39)$$

The renormalized hamiltonian for the non-relativistic delta-function potential is given by

$$H_\lambda(\mathbf{p}, \mathbf{p}') = p^2 \delta^{(2)}(\mathbf{p} - \mathbf{p}') + e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \left[\bar{V}_\lambda^{(1)}(\mathbf{p}, \mathbf{p}') + \bar{V}_\lambda^{(2)}(\mathbf{p}, \mathbf{p}') + \dots \right] , \quad (40)$$

where

$$\bar{V}_\lambda^{(1)}(\mathbf{p}, \mathbf{p}') = -\frac{\alpha_{\lambda,i}}{(2\pi)^2}, \quad (41)$$

$$\bar{V}_\lambda^{(2)}(\mathbf{p}, \mathbf{p}') = \alpha_{\lambda,i}^2 F_s^{(2)}(\mathbf{p}, \mathbf{p}'), \quad (42)$$

$$\bar{V}_\lambda^{(n)}(\mathbf{p}, \mathbf{p}') = \alpha_{\lambda,i}^n F_s^{(n)}(\mathbf{p}, \mathbf{p}'). \quad (43)$$

Here λ is a momentum cutoff related to the flow parameter by $s = 1/\lambda^4$ and i denotes the order of the calculation for the running coupling.

The renormalized hamiltonian can be used to compute eigenvalues and eigenstates. Since the hamiltonian is derived perturbatively we expect cutoff dependent errors in the observables. Formally, we can regroup the terms in the renormalized hamiltonian and write it as a momentum expansion similar to the one for the EFT in Eq. (20). The difference is that the expansion parameters are perturbative functions of the running coupling α_λ .

There are two interdependent sources of errors in the perturbative similarity renormalization group. First, α_λ is approximated and this leads to inverse-logarithmic errors. Second, the coefficients of all irrelevant operators are approximated perturbatively, and this leads to products of inverse-logarithmic and inverse-power-law errors.

In our examples we focus on the bound state errors. We fix the coupling at one scale and use the flow-equation to obtain the coupling as a function of the cutoff λ to a given order. We then perform a sequence of bound-state calculations with better approximations for the hamiltonian. Once the sources of errors are identified, it becomes relatively simple to analyze order-by-order how such errors scale with λ . In principle, to completely eliminate the errors proportional to some power m in the momentum expansion we should use the similarity hamiltonian with the exact running coupling (renormalized to all orders) and include the contributions up to $\mathcal{O}(p^m/\lambda^m)$ coming from all effective interactions (all orders in α_λ). We should emphasize that in a realistic calculation we would fit the coupling α_λ to an observable. This nonperturbative renormalization eliminates the dominant source of errors we display in SRG calculations in this article. We choose to renormalize the coupling perturbatively here because the only observable we compute is the single bound state energy of a delta-function potential, and fitting this energy would prevent us from displaying errors.

4.2 SRG results for two-dimensional δ -function

Głazek and Wilson have studied a discretized analog of the two-dimensional δ function, which allowed them to perform both perturbative and nonpertur-

bative transformation.⁵ Their perturbative results are similar to ours. In the two-dimensional case the canonical hamiltonian in momentum space with a delta-function potential can be written as

$$H(\mathbf{p}, \mathbf{p}') = h(\mathbf{p}, \mathbf{p}') + V(\mathbf{p}, \mathbf{p}') , \quad (44)$$

where $h(\mathbf{p}, \mathbf{p}') = p^2 \delta^{(2)}(\mathbf{p} - \mathbf{p}')$ corresponds to the free hamiltonian and $V(\mathbf{p}, \mathbf{p}') = -\alpha_0/(2\pi)^2$ corresponds to the Fourier transform of the delta-function potential.

Integrating out the angular variable, the flow equation obtained with Wegner's transformation in terms of matrix elements in the basis of free states is given by

$$\frac{dV_s(p, p')}{ds} = -(p^2 - p'^2)^2 V_s(p, p') - \int_0^\infty dk k (2k^2 - p^2 - p'^2) V_s(p, k) V_s(k, p') . \quad (45)$$

In principle, we can set the boundary condition at $s = 0$ (no cutoff), i.e.,

$$H_{s=0}(p, p') = H(p, p') = p^2 \delta^{(1)}(p - p') - \frac{\alpha_0}{2\pi} . \quad (46)$$

However, the hamiltonian with no cutoff produces logarithmic divergences. The boundary condition must be imposed at some other point, leading to dimensional transmutation. The reduced interaction $\bar{V}_s(p, p')$ is defined such that

$$V_s(p, p') = e^{-s(p^2 - p'^2)^2} \bar{V}_s(p, p') . \quad (47)$$

Assuming that h is cutoff independent we obtain the flow equation for the reduced interaction,

$$\begin{aligned} \frac{d\bar{V}_s}{ds} = & -e^{-2s p^2 p'^2} \int_0^\infty dk k (2k^2 - p^2 - p'^2) e^{-2s[k^4 - k^2(p^2 + p'^2)]} \\ & \times \bar{V}_s(p, k) \bar{V}_s(k, p') . \end{aligned} \quad (48)$$

This equation is solved using a perturbative expansion, starting with

$$\bar{V}_s^{(1)}(p, p') = -\frac{\alpha_s}{2\pi} . \quad (49)$$

We assume a coupling-coherent solution in the form of an expansion in powers of $\alpha_s/2\pi$, satisfying the constraint that the operators $F_s^{(n)}(p, p')$ vanish when $p = p' = 0$,

$$\bar{V}_s(p, p') = -\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left(\frac{\alpha_s}{2\pi} \right)^n F_s^{(n)}(p, p') . \quad (50)$$

Note that the expansion parameter is $\alpha_s/2\pi$.

Using the solution Eq.(50) in Eq.(48) we obtain

$$\begin{aligned} \frac{d\bar{V}_s}{ds} &= -\frac{1}{(2\pi)^2} \frac{d\alpha_s}{ds} + \sum_{n=2}^{\infty} \frac{1}{(2\pi)^n} \left[n \alpha_s^{n-1} \frac{d\alpha_s}{ds} F_s^{(n)}(p, p') + \alpha_s^n \frac{dF_s^{(n)}(p, p')}{ds} \right] \\ &= \int_0^\infty dk k (2k^2 - p^2 - p'^2) e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]} \\ &\quad \times \left[-\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left(\frac{\alpha_s}{2\pi} \right)^n F_s^{(n)}(p, k) \right] \left[-\frac{\alpha_s}{2\pi} + \sum_{m=2}^{\infty} \left(\frac{\alpha_s}{2\pi} \right)^m F_s^{(m)}(k, p') \right] \end{aligned} \quad (51)$$

This equation is solved iteratively order-by-order in $\alpha_s/2\pi$. Again, if $\alpha_s/2\pi$ is small the operator $\bar{V}_s^{(1)}(p, p')$ can be identified as the dominant term in the expansion of $\bar{V}_s(p, p')$ in powers of p and p' . This operator corresponds to a marginal operator (since the coupling is dimensionless and there is no implicit mass scale). The higher-order terms correspond to irrelevant operators.

For simplicity we use the interaction with a large cutoff on all momenta, Λ , to define the *exact* theory. We define

$$\alpha_{s_0=0} = \alpha_\Lambda = \frac{4\pi}{\ln \left(1 + \frac{\Lambda^2}{E_0} \right)}, \quad (52)$$

and set all irrelevant operators to zero at $s_0 = 0$. Note that the coupling α_{λ_0} is fixed at $\lambda_0 = \infty$ by fitting the exact binding energy. With this definition the similarity hamiltonian with no cutoff becomes well-defined and we can set all of the similarity transformation boundary conditions at $s_0 = 0$. All integrals below are cut off ($p \ll \Lambda$)

At second-order we have

$$-\frac{1}{2\pi} \frac{d\alpha_{s,\Lambda}}{ds} + \frac{1}{(2\pi)^2} \alpha_{s,\Lambda}^2 \frac{dF_{s,\Lambda}^{(2)}(p, p')}{ds} = -\alpha_{s,\Lambda}^2 I_{s,\Lambda}^{(2)}(p, p') \quad (53)$$

where

$$\begin{aligned} I_{s,\Lambda}^{(2)}(p, p') &= \frac{1}{(2\pi)^2} \int_0^\Lambda dk k (2k^2 - p^2 - p'^2) e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]} \\ &= \frac{1}{(2\pi)^2} \frac{e^{-2s p^2 p'^2}}{4s}. \end{aligned} \quad (54)$$

The resulting second-order running coupling and irrelevant operator are given respectively by

$$\alpha_{s,\Lambda,2} = \frac{\alpha_\Lambda}{1 - \frac{\alpha_\Lambda}{8\pi} [\gamma + \ln(2s\Lambda^4) - \text{Ei}(-2s\Lambda^4)]} \quad (55)$$

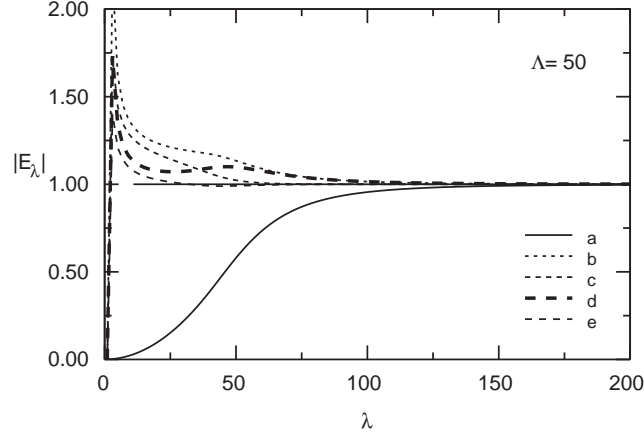


Figure 2. The SRG bound-state energy using various approximations for the similarity hamiltonian. The exact theory is fixed by choosing $\Lambda = 50$ and $E_0 = 1$.

and

$$\begin{aligned}
 F_{s,\Lambda}^{(2)}(p, p') = & \frac{1}{4} [\gamma + \ln(2s p^2 p'^2) - \text{Ei}(-2s p^2 p'^2)] \\
 & + \frac{1}{4} [\gamma + \ln(2s \Lambda^4) - \text{Ei}(-2s \Lambda^4)] \\
 & - \frac{1}{4} [\gamma + \ln(s [(p^2 - \Lambda^2)^2 + (p'^2 - \Lambda^2)^2 - (p^2 - p'^2)^2]) \\
 & \quad - \text{Ei}(-s [(p^2 - \Lambda^2)^2 + (p'^2 - \Lambda^2)^2 - (p^2 - p'^2)^2])] . \quad (56)
 \end{aligned}$$

In Fig. 2 we show the binding-energy as a function of the cutoff and in Fig. 3 we show the errors in the binding-energy obtained using the following approximations for the potential with $\Lambda = 50$:

(a) marginal operator with coupling (α_{λ_0}) ,

$$V_{\lambda,\Lambda}(p, p') = -\frac{\alpha_{\Lambda}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} ; \quad (57)$$

(b) marginal operator with running coupling renormalized to second-order $(\alpha_{\lambda,\Lambda,2})$,

$$V_{\lambda,\Lambda}(p, p') = -\frac{\alpha_{\lambda,\Lambda,2}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} ; \quad (58)$$

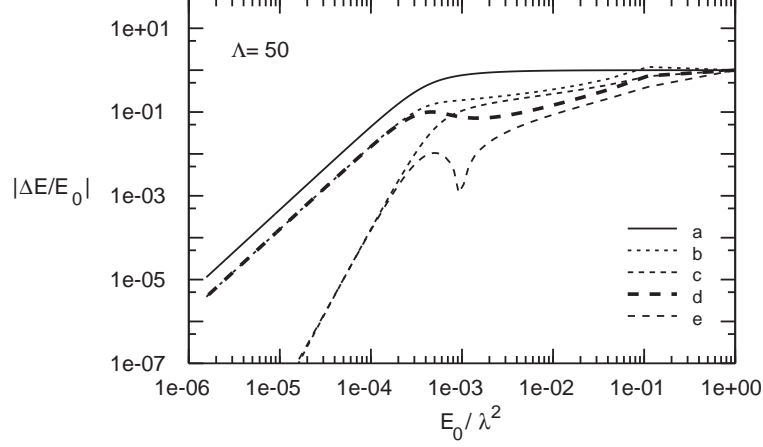


Figure 3. The SRG errors in the binding energy using various approximations for the similarity hamiltonian. The exact theory is fixed by choosing $\Lambda = 50$ and $E_0 = 1$.

(c) marginal operator plus second-order irrelevant operator with running coupling renormalized to second-order ($\alpha_{\lambda,\Lambda,2}, F_{\lambda}^{(2)}$),

$$V_{\lambda,\Lambda}(p, p') = \left[-\frac{\alpha_{\lambda,\Lambda,2}}{2\pi} + \left(\frac{\alpha_{\lambda,\Lambda,2}}{2\pi} \right)^2 F_{\lambda,\Lambda}^{(2)}(p, p') \right] e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}}; \quad (59)$$

(d) marginal operator with running coupling renormalized to third-order ($\alpha_{\lambda,\Lambda,3}$),

$$V_{\lambda,\Lambda}(p, p') = -\frac{\alpha_{\lambda,\Lambda,3}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}}; \quad (60)$$

(e) marginal operator plus second-order irrelevant operator with running coupling renormalized to third-order ($\alpha_{\lambda,\Lambda,3}, F_{\lambda}^{(2)}$),

$$V_{\lambda,\Lambda}(p, p') = \left[-\frac{\alpha_{\lambda,\Lambda,3}}{2\pi} + \left(\frac{\alpha_{\lambda,\Lambda,3}}{2\pi} \right)^2 F_{\lambda,\Lambda}^{(2)}(p, p') \right] e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}}. \quad (61)$$

In Fig. 2 we clearly see all of the binding energies approaching the exact result, which we choose to be 1. If the coupling does not run (a) the energy falls to zero as the cutoff is lowered. When the coupling is allowed to run to second order (b) the binding energy deviates from the exact result more slowly as the

cutoff is lowered, and subsequent improved approximations further remove dependence on λ .

Fig. 3 is a log-log plot of the relative error in the binding energy as a function of E_0/λ^2 . When $\lambda \gg \Lambda$ the errors scale like inverse powers of E_0/λ^2 . They are forced to approach the correct result as $\lambda \rightarrow \infty$, and the exact running coupling deviates from the approximate running coupling as a power of Λ/λ in this region, even when the approximate coupling is not allowed to run. The coefficients of these powers are large, and the analysis changes as $\lambda \rightarrow \Lambda$. The errors become logarithmic for (a), but they scale as an inverse-logarithm squared in (b) where the coupling runs to second-order. This removes the dominant inverse-logarithmic errors, and we can clearly see further improvement when the coupling runs to third-order in (d). The addition of irrelevant operators in (c) and (e) are important, but the third-order error in the running coupling becomes as large as the error from the leading irrelevant operators, which shows up when (d) crosses (c). When the second-order irrelevant operators are added with the coupling running to third order, we obtain approximate power-law improvement over (b).

5 Conclusion

Renormalization not only removes unphysical divergences, it also allows us to work with finite cutoffs while controlling the errors they introduce. Finite cutoffs are often required for nonperturbative calculations, and the precise control of the errors they introduce is essential.

In light-front QCD we can use the SRG to compute the hamiltonian order-by-order in the canonical QCD coupling, α_λ . We are free to choose λ so that multi-parton high-energy states effectively decouple from few-parton low-energy states, as long as this happens before α_λ becomes too large. We adjust the coupling non-perturbatively by fitting low-energy data. This should remove all inverse-logarithmic errors, leaving only powers of inverse-logarithms and inverse-powers of the cutoff. We can systematically lower the errors by adding irrelevant operators, all of which are fixed by coupling coherence.

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